Student Number ....

Barker College

2007
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

Staff Involved:

- WMD*
- JM
- VAB
- BHC

AM TUESDAY 31 JULY

30 copies

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- Write your Barker Student Number on ALL pages of your answer sheets
- A table of standard integrals is provided on page 12

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value
- ALL necessary working should be shown in every question
- Start each question on a NEW page
- Write on one side of each answer page
- Marks may be deducted for careless or badly arranged work

Total marks – 120
Attempt Questions 1 – 8
ALL questions are of equal value
Answer each question on a SEPARATE sheet of paper

Marks

Question 1 (15 marks) **[START A NEW PAGE]**

(a) Find $\int \frac{(\ln x)^5}{x}\,dx$. **1**

(b) Use the table of standard integrals to help find $\int \frac{dy}{\sqrt{4y^2+36}}$. **1**

(c) Let $\frac{2x+2}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

(i) Find the value of A, B and C. **2**

(ii) Hence, find $\int \frac{2x+2}{(x-1)(x^2+1)}\,dx$. **2**

(d) (i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

(ii) Hence or otherwise evaluate $\int_0^1 x^2\sqrt{1-x}\,dx$. **3**

(e) Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$. **4**

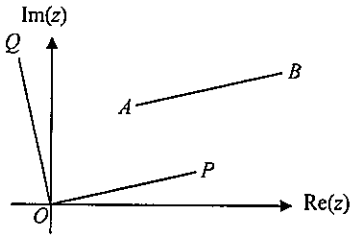
Marks

Question 2 (15 marks) **[START A NEW PAGE]**

(a) Find $\sqrt{21+20i}$ in the form $x+iy$. **3**

(b) Sketch the locus of z described by the inequality $|z-1+i|\leq 1$ and state the minimum value of $\arg z$. **2**

(c)



In the Argand diagram above, intervals AB, OP and OQ are equal in length, OP is parallel to AB and $\angle POQ=\frac{\pi}{2}$.

(i) If A and B represent the complex numbers $3+5i$ and $9+8i$ respectively, find the complex number which is represented by P. **1**

(ii) Hence find the complex number which is represented by Q. **1**

(d) If $z=x+iy\ (x,y\in R)$, find and describe in words, the locus of the points $P(x,y)$ such that $\operatorname{Im}\left(z+\frac{1}{z}\right)=0$. **4**

(e) Write $\sqrt{3}+i$ and $\sqrt{3}-i$ in modulus/argument form. Hence show that $\left(\sqrt{3}+i\right)^{10}+\left(\sqrt{3}-i\right)^{10}$ is a rational number. **4**

Marks

Question 3 (15 marks) **[START A NEW PAGE]**

(a) The equation $x^3-3x^2+6x-4=0$ has roots α,β and γ. Find α,β and γ, given that one root is $x=1+i\sqrt{3}$. **2**

(b) (i) Prove that if polynomial $P(x)$ has a root of multiplicity m at $x=c$ then $P'(x)$ has a root of multiplicity $(m-1)$ at the same point. **2**

(ii) The equation $8x^3+4x^2=2x+1$ has a double root and a single root. Find the double root. **3**

(c) If α,β,γ are the roots of $x^3-x-1=0$, find the equation with roots $\frac{1}{1-\alpha},\frac{1}{1-\beta},\frac{1}{1-\gamma}$. Hence state the value of $\frac{1}{1-\alpha}+\frac{1}{1-\beta}+\frac{1}{1-\gamma}$. **4**

(d) If one of the roots of the equation $x^3+ax^2+bx+c=0$ is the sum of the other two roots show that $a^3-4ab+8c=0$ **4**

Marks

Question 4 (15 marks) **[START A NEW PAGE]**

(a) ϕ is a complex cube root of unity, i.e. $\phi^3 = 1$, $\phi \neq 1$.

(i) Find the value of $\phi + \phi^2$ **1**

(ii) Prove that $(a-b)(a-\phi b)(a-\phi^2 b) = a^3 - b^3$ **2**

(b) (i) Sketch $y = \dfrac{x-2}{x+1}$. Show all points of intersection with the coordinate axes and any asymptotes. **2**

(ii) Use your graph in (i) to sketch:

(α) $y = \left(\dfrac{x-2}{x+1}\right)^2$ **2**

(β) $y^2 = \left(\dfrac{x-2}{x+1}\right)$ **2**

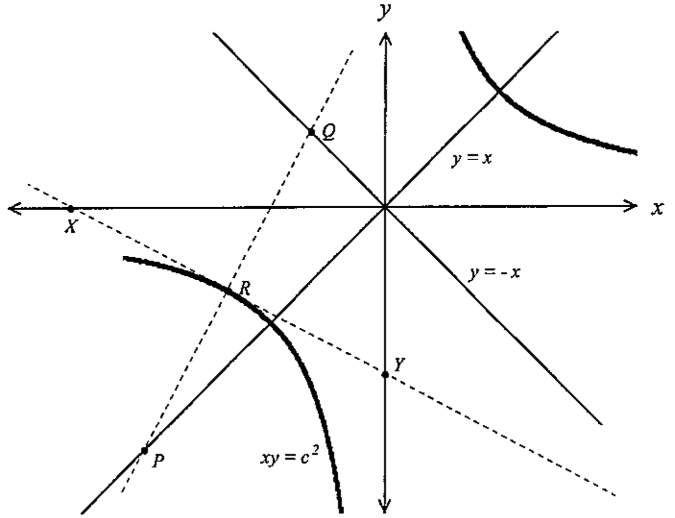
(c) By considering the behaviour of $y = \cos^{-1} x$ and $u = e^x$, or otherwise, draw the graph of $y = \cos^{-1}\left(e^x\right)$. Calculus need not be used. **3**

(d) Sketch, without using calculus, $y = x|x-2|$. Hence, or otherwise, state for what values of k, $x|x-2| = k$ has 3 solutions. **3**

Marks

Question 5 (15 marks) **[START A NEW PAGE]**

(a)



In the diagram above $R(ct, \frac{c}{t})$ is a point on the rectangular hyperbola $xy = c^2$.

The tangent to the hyperbola at R meets the x-axis at X and the y-axis at Y.
The normal to the hyperbola at R meets the line $y = x$ at P and the line $y = -x$ at Q.
You are given that the equation of the tangent at R is $x + t^2 y = 2ct$.

(i) Prove that the equation of the normal at R is $ty + ct^4 = t^3 x + c$. **2**

(ii) It can be shown that P is the point $\left(\dfrac{c\left(t^2+1\right)}{t}, \dfrac{c\left(t^2+1\right)}{t}\right)$. **3**

Find the coordinates of X and Y. Prove that Q is the point $\left(\dfrac{c\left(t^2-1\right)}{t}, \dfrac{c\left(1-t^2\right)}{t}\right)$.

(iii) Show that PQ and XY bisect each other. **2**

(iv) Show that $PQ = XY$. **2**

(v) What type of quadrilateral is $XQYP$? Justify your answer. **1**

Question 5 continues on page 7

Marks

Question 5 (continued)

(b) A tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $P(a\sec\theta, b\tan\theta)$ meets the nearest directrix at Q. If S is the nearest focus, prove that PQ subtends a right angle at S, i.e. $\angle PSQ = 90°$. **5**

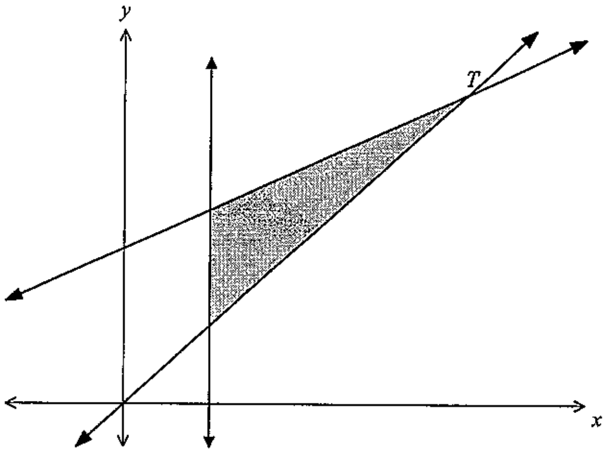
You may assume that the equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

Marks

Question 6 (15 marks) **[START A NEW PAGE]**

(a) Find the Cartesian equation of the ellipse $x=1+3\cos\theta$, $y=2\sin\theta-2$. **2**

(b) The area bounded by the lines $2x-y=0$, $x-y+4=0$ and $x=1$, shown below, is to be rotated about the y-axis. **4**



Show that the x-coordinate of T is 4. Then, using the method of cylindrical shells, show that the volume of the solid of revolution generated is 18π cubic units.

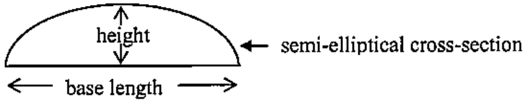
Question 6 continues on page 9

Question 6 (continued)

(c) (i) Explain why $\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{\pi a^2}{4}$. It is not necessary to use formal integration. **1**

(ii) Using (i) and symmetry, or otherwise, show that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab units^2. **3**

(iii) The base of a particular solid is the circle $x^2 + y^2 = r^2$. Every cross-section perpendicular to the x-axis is a semi-ellipse. The base length of each semi-elliptical cross-section is 4 times its height. **5**



Use part (ii) to find an expression for the volume of a cross-sectional slice and hence find the volume of the solid.

Question 7 (15 marks) **[START A NEW PAGE]**

(a) How many positive integers n are there such that $n + 3$ divides $n^2 + 7$ without a remainder? **3**

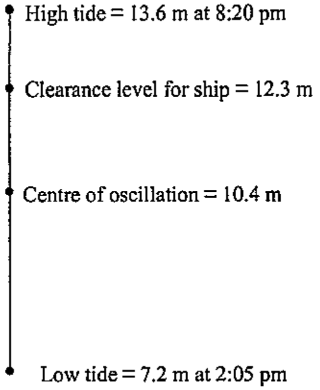
(b) Four families each have four children. What is the probability that exactly two of the families have two boys and two girls? **3**

(c) From the letters of the word RENEGADE, three are taken at random and placed in a line.

(i) How many different 3 letter sequences are there with exactly one E in the sequence? **1**

(ii) How many different 3 letter sequences are there altogether? **3**

(d) The depth of the water in a harbour is 7.2 m at low water and 13.6 m at high water. On Monday, low water is at 2:05 pm and high water is at 8:20 pm. A particular ship requires water to a depth of at least 12.3 m to leave harbour. The ship's captain wants to leave harbour between noon and midnight. Find between what times the ship can leave, assuming that the motion of the tide is simple harmonic. **5**



Marks

Question 8 (15 marks) **[START A NEW PAGE]**

(a) (i) Use De Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$. **2**

(ii) Use the result to solve $8x^3 - 6x + 1 = 0$. **3**

(iii) Hence deduce that $\sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = 8$. **2**

(b) (i) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$ then show that $I_n = \frac{n-1}{n} I_{n-2}$. **4**

(ii) By first finding I_5, I_7 and I_9 in unsimplified form, **4**

and noting a pattern in your answers,

show that $I_n = \frac{\left\{\left[\frac{1}{2}(n-1)\right]!\right\}^2}{n!} 2^{n-1}$ if n is odd.

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x > 0$

Barker College

Extension 2 Mathematics Trial HSC Exam Solutions 2007

Question 1

(a) $\int \frac{(\ln x)^5}{x} = \int (\ln x)^5 \, d(\ln x)$

$= \underline{\underline{\frac{(\ln x)^6}{6} + c}}$

(b) $\int \frac{dy}{\sqrt{4y^2+36}} = \frac{1}{2}\int \frac{dy}{\sqrt{y^2+3^2}}$

$= \underline{\underline{\frac{1}{2} \ln \left\{ y + \sqrt{y^2+9} \right\} + c}}$

(c) (i) $2x + 2 \equiv A(x^2+1) + (Bx+C)(x-1)$

when $x = 1$, $4 = 2A \Rightarrow A = 2$

comparing coefficients of x^2 : $0 = A + B$

$0 = 2 + B$

$\Rightarrow B = -2$

when $x = 0$, $2 = 2 + C\times -1 \Rightarrow C = 0$.

$\therefore \underline{\underline{A = 2,\ B = -2,\ C = 0}}$

(ii) $\int \frac{2x+2}{(x-1)(x^2+1)} dx = \int \left\{ \frac{2}{x-1} - \frac{2x}{x^2+1} \right\} dx$

$= \underline{\underline{2\ln|x-1| - \ln(x^2+1) + c}}$

(d) (i) $\int_{x=0}^{x=a} f(a-x)\, dx = -\int_{u=a}^{u=0} f(u)\, du$ let $u = a - x$ $\therefore du = -dx$

$= \int_0^a f(u)\, du$

$= \int^a f(x)\, dx$

(ii) $\int_0^1 x^2(1-x)^{\frac{1}{2}} dx$

$= \int_0^1 (1-x)^2 x^{\frac{1}{2}} dx$

$= \int_0^1 x^{\frac{1}{2}}(1 - 2x + x^2)\, dx$

$= \int_0^1 (x^{\frac{1}{2}} - 2x^{\frac{3}{2}} + x^{\frac{5}{2}})\, dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}} - 2\times\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}}\right]_0^1$

$= \frac{2}{3} - \frac{4}{5} + \frac{2}{7}$

$= \underline{\underline{\frac{16}{105}}}$

(e) $\cos x = \cos\left(\frac{x}{2} + \frac{x}{2}\right)$

$= 2\cos^2\frac{x}{2} - 1$

$\therefore \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{2\cos^2\frac{x}{2}} = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sec^2\frac{x}{2}\, dx$

$= \cancel{\frac{1}{2}} \left[\cancel{2}\tan\frac{x}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \tan\frac{\pi}{4} - \tan\frac{\pi}{6}$

$= 1 - \frac{1}{\sqrt{3}}$

$= \frac{1}{3}(3 - \sqrt{3})$

or

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{\frac{2}{t^2+1}}{1+\frac{1-t^2}{1+t^2}}\, dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{1+t^2+1-t^2}\, dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} dt$$

$$= \left[\, t \,\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= 1 - \frac{1}{\sqrt{3}}$$

$$= \frac{1}{3}\left(3-\sqrt{3}\right)$$

Let $\cos x = \dfrac{1-t^2}{1+t^2}$

$$t = \tan\frac{x}{2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \frac{1}{2}\left(\tan^2\frac{x}{2}+1\right) = \frac{1}{2}(t^2+1)$$

$$\therefore\ dx = \frac{2\,dt}{t^2+1}$$

$x = \frac{\pi}{3},\ t = \frac{1}{\sqrt{3}}\ ;\ x = \frac{\pi}{2},\ t = 1$

Question 2

(a) $21 + 20i = (x^2-y^2) + 2xyi$

$x^2 - y^2 = 21 \quad \dots (1)$

$2xy = 20 \Rightarrow y = \frac{10}{x} \quad \dots (2)$

sub (2) in (1) : $x^2 - \frac{100}{x^2} = 21$

$$x^4 - 21x^2 - 100 = 0$$

$$(x^2-25)(x^2+4) = 0$$

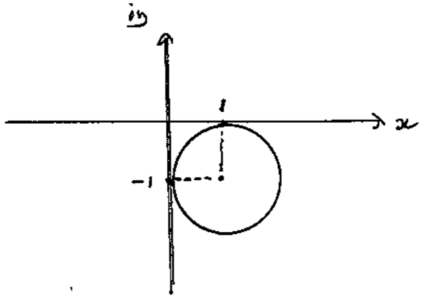
$$\therefore\ x = \pm 5$$

$$x = 5 \Rightarrow y = 2$$

$$x = -5 \Rightarrow y = -2$$

$$\therefore\ \sqrt{21+20i} = \pm(5+2i)$$

(b)



min. arg $z = -\frac{\pi}{2}$

(c) (i) $P = (9+8i) - (3+5i)$

$= 6 + 3i$

(ii) $Q = i(6+3i)$

$= -3 + 6i$

(d) $z + \frac{1}{z} = x + iy + \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

$$= x + iy + \frac{x-iy}{x^2+y^2}$$

$$= \frac{x(x^2+y^2) + iy(x^2+y^2) + x - iy}{x^2+y^2}$$

$$= \frac{x(x^2+y^2+1) + iy(x^2+y^2-1)}{x^2+y^2}$$

$$Im\left(z+\frac{1}{z}\right) = \frac{y(x^2+y^2-1)}{x^2+y^2} = 0$$

$\therefore\ y=0$ or $x^2+y^2=1$, $x^2+y^2\neq 0$

The locus of P is the unit circle and the x-axis except for the origin.

(e) $\sqrt{3}+i = 2\text{ cis}\frac{\pi}{6}$, $\sqrt{3}-i = 2\text{ cis}\left(-\frac{\pi}{6}\right)$

$$\therefore (\sqrt{3}+i)^{10}+(\sqrt{3}-i)^{10}$$

$$= \left(2\text{ cis}\frac{\pi}{6}\right)^{10} + \left(2\text{ cis}\left(\frac{-\pi}{6}\right)\right)^{10}$$

$$= 2^{10}\text{ cis}\frac{10\pi}{6} + 2^{10}\text{ cis}\left(\frac{-10\pi}{6}\right)$$

$$= 1024\left\{\text{cis}\left(-\frac{\pi}{3}\right)+\text{cis}\left(\frac{\pi}{3}\right)\right\}$$

$$= 1024\left\{\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)+\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right\}$$

$$= 1024\left\{\cos\frac{\pi}{3}\cancel{-i\sin\frac{\pi}{3}}+\cos\frac{\pi}{3}\cancel{+i\sin\frac{\pi}{3}}\right\}$$

$$= 1024\left\{\frac{1}{2}+\frac{1}{2}\right\}$$

$= 1024$ which is a rational number.

Question 3

(a) $x^3-3x^2+6x-4=0$ has real coefficients so $1-i\sqrt{3}$ is also a root.

Let $\alpha = 1+i\sqrt{3}$ and $\beta = 1-i\sqrt{3}$

$\alpha+\beta+\gamma = 3$

$\therefore (1+i\sqrt{3})+(1-i\sqrt{3})+\gamma = 3$

$\Rightarrow \gamma = \underline{\underline{1}}$

(b) (i) Let $P(x) = (x-c)^m Q(x)$

$\therefore P'(x) = m(x-c)^{m-1}Q(x) + (x-c)^m Q'(x)$

$= (x-c)^{m-1}\{mQ(x)+(x-c)Q'(x)\}$

$\Rightarrow P'(x)$ has a root of multiplicity $(m-1)$ at $x=c$.

(ii) Let $P(x) = 8x^3+4x^2-2x-1$

$P'(x) = 24x^2+8x-2$

$P'(x)=0 \Rightarrow 12x^2+4x-1=0$

$(6x-1)(2x+1)=0$

$x=\frac{1}{6}, -\frac{1}{2}$

$P\left(\frac{1}{6}\right) = \frac{8}{216}+\frac{4}{36}-\frac{2}{6}-1 \neq 0$

$P\left(-\frac{1}{2}\right) = -1+1+1-1 = 0$

$\therefore\ \underline{\underline{x=-\frac{1}{2} \text{ is the double root}}}$.

(c) Put $y=\frac{1}{1-x}$

$1-x = \frac{1}{y}$

$x = 1-\frac{1}{y}$

$x = \frac{y-1}{y}$

Req'd eq'n is $\left(\frac{y-1}{y}\right)^3 - \left(\frac{y-1}{y}\right) - 1 = 0$

$\therefore (y-1)^3 - y^2(y-1) - y^3 = 0$

$\therefore \quad \cancel{y^3} - 3y^2 + 3y - 1 - \cancel{y^3} + y^2 - y^3 = 0$

$\therefore \quad -y^3 - 2y^2 + 3y - 1 = 0$

i.e. $\quad y^3 + 2y^2 - 3y + 1 = 0$

$\therefore \quad \sum \frac{1}{1-\alpha} = -2$

(d) Let the roots be α, β, γ where $\alpha = \beta + \gamma$

$\sum$ roots : $\alpha + \beta + \gamma = -a$

$\therefore \quad 2\alpha = -a \Rightarrow \alpha = -\frac{a}{2}$

Sum of roots 2 at a time : $\alpha\beta + \alpha\gamma + \beta\gamma = b$

$\therefore \quad \alpha(\beta+\gamma) + \beta\gamma = b$

$\therefore \quad \left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right) + \beta\gamma = b$

$\therefore \quad \beta\gamma = -\frac{a^2}{4} + b \quad \ldots (1)$

Product of roots : $\alpha\beta\gamma = -c$

$\therefore \quad \beta\gamma = -\frac{c}{\alpha}$

$\therefore \quad \beta\gamma = \frac{2c}{a} \quad \ldots (2)$

from (1) & (2) : $-\frac{a^2}{4} + b = \frac{2c}{a}$

$\therefore \quad -a^3 + 4ab = 8c$

$\therefore \quad 0 = \underline{\underline{a^3 - 4ab + 8c}}$

Question 4

(a) (i) $\phi^3 = 1, \; \phi \neq 1$

$\phi^3 - 1 = 0$

$(\phi - 1)(\phi^2 + \phi + 1) = 0$

$\therefore \phi^2 + \phi + 1 = 0 \quad (\phi \neq 1)$

$\therefore \underline{\underline{\phi^2 + \phi = -1}}$

(ii) $(a-b)(a-\phi b)(a-\phi^2 b)$

$= (a-b)(a^2 - \phi^2 ab - \phi ab + \phi^3 b^2)$

$= (a-b)(a^2 + b^2 - ab(\phi^2 + \phi))$

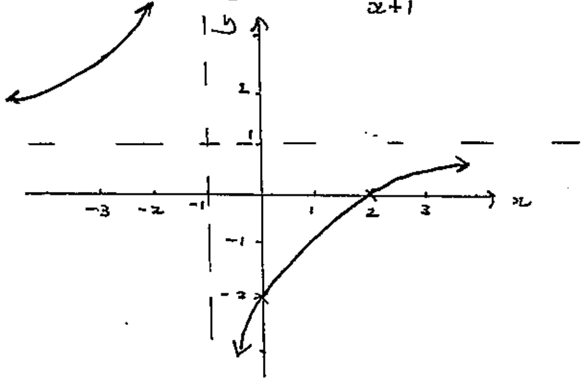
$= (a-b)(a^2 + b^2 + ab)$

$= \underline{\underline{a^3 - b^3}}$

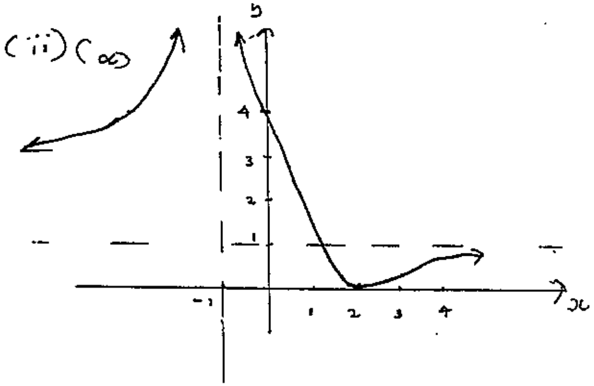
(b) (i) $y = \frac{(x+1) - 3}{x+1} = 1 - \frac{3}{x+1}$

$x = 0, \; y = -2$

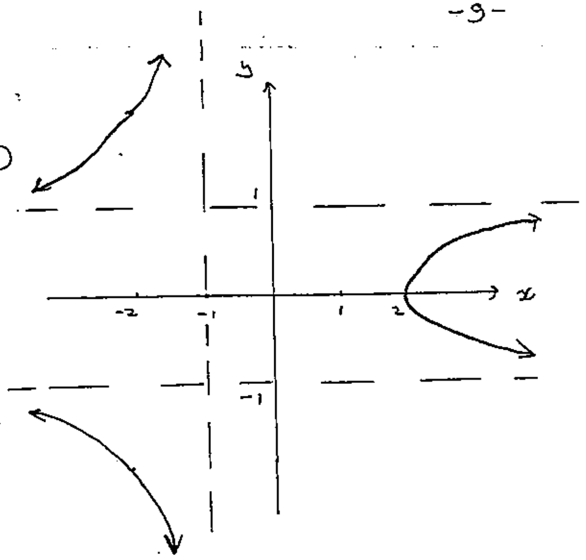
$y = 0, \; x = 2$



(ii) (α)



(β)



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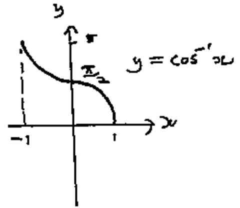
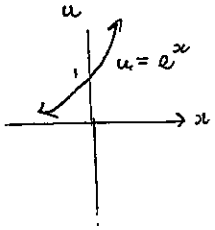
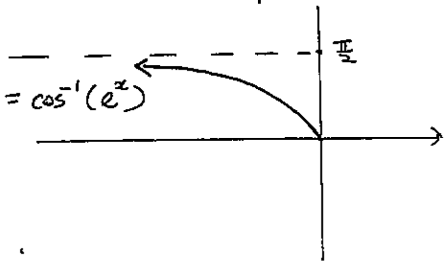
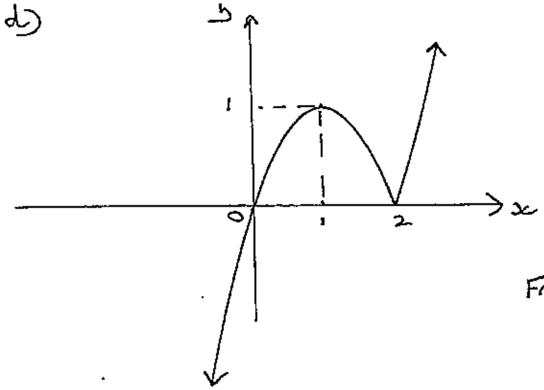






(d)



When $x = 1$, $y = 1 \times |1-2| = 1$

For what values of k will $y = k$ intersect $y = x|x-2|$ in 3 places?

From the graph, $0 < k < 1$.

Question 5

(a) (i) $y = \frac{c^2}{x}$

$\therefore \frac{dy}{dx} = -\frac{c^2}{x^2}$

when $x = ct$, $\frac{dy}{dx} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2} \Rightarrow m_N = t^2$

$\therefore$ Eq'n of normal : $y - \frac{c}{t} = t^2(x - ct)$

$$ty - c = t^3x - ct^4$$

$$ty + ct^4 = t^3x + c$$

(ii) Tangent is $x + t^2y = 2ct$.

when $x = 0$, $t^2y = 2ct$

$y = \frac{2c}{t} \Rightarrow Y$ is $(0, \frac{2c}{t})$

when $y = 0$, $x = 2ct \Rightarrow X$ is $(2ct, 0)$

for Q, sub $y = -x$ in $ty + ct^4 = t^3x + c$:

$$-xt + ct^4 = t^3x + c$$

$$x(t^3+t) = c(t^4-1)$$

$$x = \frac{c(t^2-1)(t^2+1)}{t(t^2+1)} = \frac{c(t^2-1)}{t}$$

sub $x = \frac{c(t^2-1)}{t}$ in $y = -x$

$\therefore y = \frac{c(1-t^2)}{t}$

$\Rightarrow Q$ is $\left(\frac{c(t^2-1)}{t}, \frac{c(1-t^2)}{t}\right)$

(iii) Midpoint of PQ $= \left(\dfrac{\frac{c}{t}(t^2+1+t^2-1)}{2} , \dfrac{c}{t}\dfrac{(t^2+1+1-t^2)}{2} \right)$

$$= \left(\frac{c}{2t}(2t^2) , \frac{c}{2t} \times 2 \right)$$

$$= \left(ct , \frac{c}{t} \right)$$

$$= R$$

midpoint of XY $= \left(\dfrac{2ct+0}{2} , \dfrac{0+\frac{2c}{t}}{2} \right)$

$$= \left(ct, \frac{c}{t} \right)$$

$$= R$$

XY & PQ share the same midpoint, hence they must bisect each other.

(iv) $XY = \sqrt{(2ct)^2 + \left(\frac{2c}{t}\right)^2}$

$$= \sqrt{4c^2t^2 + \frac{4c^2}{t^2}}$$

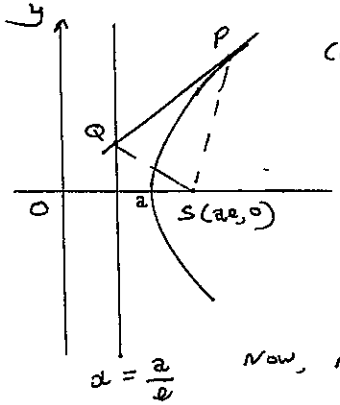
$PQ = \sqrt{\left(\frac{c}{t}(t^2+1-(t^2-1))\right)^2 + \left(\frac{c}{t}(t^2+1-(1-t^2))\right)^2}$

$$= \sqrt{\frac{4c^2}{t^2} + \frac{c^2}{t^2} \times 4t^4}$$

$\therefore \quad PQ = \sqrt{\frac{4c^2}{t^2} + 4c^2t^2} = XY$

(v) $\because$ The diagonals bisect each other at right angles and are equal in length, $XQYP$ is a square.

(b)



Coordinates of Q:

when $x = \frac{a}{e}$, $\frac{\sec\theta}{e} - \frac{y\tan\theta}{b} = 1$

$\therefore \quad b\sec\theta - ey\tan\theta = be$

$$\Rightarrow \quad y = \frac{b(\sec\theta - e)}{e\tan\theta}$$

Now, $m_{QS} = \dfrac{\frac{b(\sec\theta - e)}{e\tan\theta}}{\frac{a}{e} - ae}$

$$= \frac{b(\sec\theta - e)}{a\tan\theta(1-e^2)}$$

and $m_{PS} = \dfrac{b\tan\theta}{a\sec\theta - ae} = \dfrac{b\tan\theta}{a(\sec\theta - e)}$

$m_{PS} \times m_{QS} = \dfrac{b(\sec\theta - e)}{a\tan\theta(1-e^2)} \times \dfrac{b\tan\theta}{a(\sec\theta - e)}$

$$= \frac{b^2}{a^2(1-e^2)}$$

i.e. $m_{PS} \times m_{QS} = -1$ as $b^2 = a^2(e^2-1)$

$\Rightarrow PS \perp QS$.

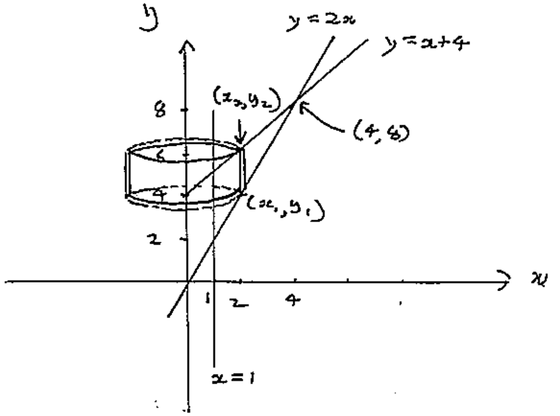
Question 6

(a) $\cos\theta = \frac{x-1}{3}$, $\sin\theta = \frac{y+2}{2}$

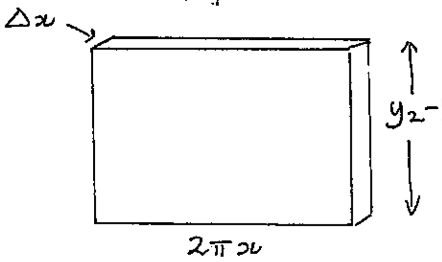
$$\therefore \frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = \cos^2\theta + \sin^2\theta$$

$$\therefore \frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$$

(b)



$\left.\begin{array}{l} y=2x \\ y=x+4 \end{array}\right\} \Rightarrow x=4, y=8$



$y_2 - y_1 = x+4-2x$

$= 4-x$

$\Delta V \doteqdot 2\pi x(4-x)\Delta x$

$$V = 2\pi\int_{x=1}^{x=4} x(4-x)\,dx$$

$$= 2\pi\int_1^4 (4x - x^2)\,dx$$

$$= 2\pi\left[2x^2 - \frac{x^3}{3}\right]_1^4$$

$$= 2\pi\left[\left(32 - \frac{64}{3}\right) - \left(2 - \frac{1}{3}\right)\right]$$

$$= 18\pi \ u^3$$

OR

$$\Delta V = \pi\left[(x+\Delta x)^2 - x^2\right]\left[4-x\right]$$

$$= \pi\left[2x\Delta x + (\Delta x)^2\right]\left[4-x\right]$$

$$V = \lim_{\Delta x\to 0}\sum_{x=1}^{4}\pi\left[2x\Delta x + (\Delta x)^2\right]\left[4-x\right]$$

$$= \lim_{\Delta x\to 0}\sum_{x=1}^{4} 2\pi\left(x + \frac{\Delta x}{2}\right)(4-x)\Delta x$$

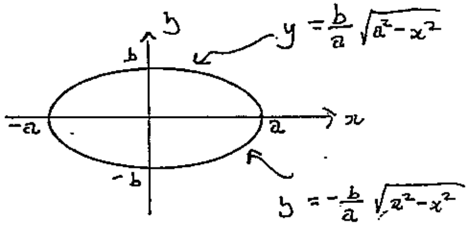
(c)

(i) $\int_0^a \sqrt{a^2-x^2}\,dx$ calculates the area of the 1st quadrant of a circle centred at the origin with radius a

$$\therefore \int_0^a \sqrt{a^2-x^2}\,dx = \frac{1}{4}\times\pi\times a^2.$$

$$= \frac{\pi a^2}{4}$$

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$= \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

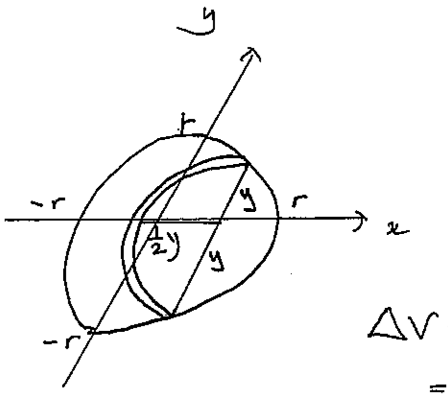
$$A = \frac{2b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$$

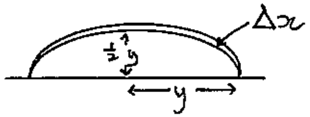
$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\times\frac{\pi a^2}{4}$$

$$= \pi ab \ u^2$$

(iii)





$$\Delta V = \tfrac{1}{2} \times \pi \times y \times \tfrac{1}{2} y \times \Delta x$$

$$= \frac{\pi}{4} y^2 \Delta x$$

Now $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

$$\therefore V = \frac{\pi}{4} \int_{-r}^{r} (r^2 - x^2)\, dx$$

$$= \frac{\pi}{2} \int_{0}^{r} (r^2 - x^2)\, dx$$

$$= \frac{\pi}{2} \left[r^2 x - \frac{x^3}{3} \right]_0^r$$

$$= \frac{\pi}{2} \left\{ r^3 - \frac{r^3}{3} \right\}$$

$$= \frac{\pi}{2} \times \frac{2r^3}{3}$$

$$= \frac{\pi r^3}{3} \text{ units}^3$$

Question 7

(a)

$$\begin{array}{r|l} & n-3 \\ \hline n+3 & n^2+0n+7 \\ & n^2+3n \\ \hline & \quad -3n+7 \\ & \quad -3n-9 \\ \hline & \qquad 16 \end{array}$$

$\therefore n^2+7 = (n+3)(n-3)+16$

$\therefore n+3 \mid n^2+7 \Rightarrow n+3 \mid 16$

$n+3 \mid 16$ if $n = 1, 5, 13$

∴ There are 3 such positive integers.

(b)

Prob. of 2 girls & 2 boys in 1 family $= {}^4C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^2$

$= \frac{3}{8}$

Req'd prob. $= {}^4C_2 \left(\frac{3}{8}\right)^2 \left(\frac{5}{8}\right)^2$

$= \frac{6 \times 9 \times 25}{8^4}$

$= \frac{675}{2048}$

(c) (i) Req'd no. $= {}^5C_2 \times 3!$

$= 10 \times 6$

$= 60$

(ii) No. with no E's $= {}^5C_3 \times 3!$ or $5 \times 4 \times 3 = 60$

$= 10 \times 6$

$= 60$

No. with 1 E = 60 from (i)

No. with 2 E's = $= {}^5C_1 \times 3$

$= 5 \times 3$

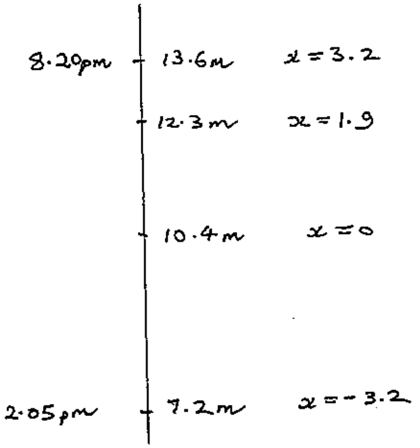
$= 15$

No. with 3 E's = 1

Total no. $= 60 + 60 + 15 + 1$

$= 136$

(d)



Let $x = A\cos(nt+\alpha)$

where t is time in minutes.

$\therefore \quad x = 3.2\cos\left(\frac{\pi t}{375}+\alpha\right)$

$A = \frac{13.6-7.2}{2} = 3.2$

$n = \frac{2\pi}{T}$, $T = 375\times 2$

$= 750$

$\therefore \quad n = \frac{\pi}{375}$

when $t=0$, $x = -3.2$

$-3.2 = 3.2\cos\alpha$

$\cos\alpha = -1$

$\alpha = \pi$

$\therefore \quad x = 3.2\cos\left(\frac{\pi t}{375}+\pi\right)$

$= -3.2\cos\left(\frac{\pi t}{375}\right)$

when $x = 1.9$, $1.9 = -3.2\cos\left(\frac{\pi t}{375}\right)$

$\cos\left(\frac{\pi t}{375}\right) = -\frac{1.9}{3.2}$

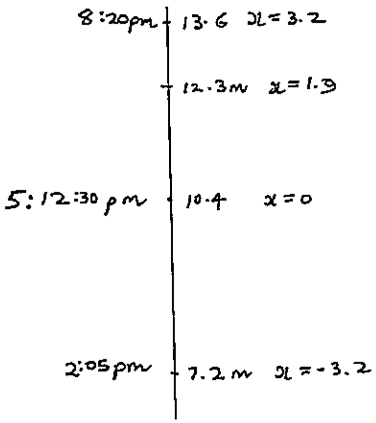
$\frac{\pi t}{375} = \ldots, \underset{\times}{-\cos^{-1}\left(\frac{1.9}{3.2}\right)}, \underset{\uparrow}{\pi - \cos^{-1}\left(\frac{1.9}{3.2}\right)}, \underset{\uparrow}{\pi + \cos^{-1}\left(\frac{1.9}{3.2}\right)}, \ldots$

$t = ---, 263.38\ldots, 486.61\ldots$

$t = 264$ (rounding up), 486 (rounding down)

∴ He should leave between 6.29 pm and 10.11 pm.

OR



Let $x = A\sin nt$,

taking $t=0$ when $x=0$.

$A = 3.2$, $n = \frac{2\pi}{T}$, $T = 375\times 2$

$= \frac{\pi}{375}$ $\quad = 750$

$\therefore \quad x = 3.2\sin\left(\frac{\pi t}{375}\right)$

when $x = 1.9$,

$1.9 = 3.2\sin\left(\frac{\pi t}{375}\right)$

$\frac{\pi t}{375} = \sin^{-1}\left(\frac{1.9}{3.2}\right), \pi - \sin^{-1}\left(\frac{1.9}{3.2}\right), \ldots$

$t = 75.88\ldots, 299.11\ldots$

∴ He should leave between 6:29 pm and 10:11 pm.

↑ rounded up ↑ rounded down

Question 8

(a) (i) $(\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

By De Moivre's theorem $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$\therefore \cos 3\theta + i\sin 3\theta = \cos^3\theta - 3\cos\theta\sin^3\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

Equating real parts :

$$\cos 3\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$$
$$= 4\cos^3\theta - 3\cos\theta$$

(ii) $8x^3 - 6x + 1 = 0$

$8x^3 - 6x = -1$

$4x^3 - 3x = -\frac{1}{2}$

$4\cos^3\theta - 3\cos\theta = -\frac{1}{2}$, where $x = \cos\theta$

$\therefore \cos 3\theta = -\frac{1}{2}$ { from (i) }

$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

$\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

$\therefore x = \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$

(iii) Product of roots $= -\frac{1}{8}$

$\therefore \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} = -\frac{1}{8}$

$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\left(-\cos\frac{\pi}{9}\right) = -\frac{1}{8}$

$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$

$\frac{1}{\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}} = 8$

$\therefore \sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = \underline{\underline{8}}$

(b) (i) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$= \int_0^{\frac{\pi}{2}} \sin^{n-1}x \; d(-\cos x)$

$= \left[-\cos x \sin^{n-1}x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \; d(\sin^{n-1}x)$

$= 0 + \int_0^{\frac{\pi}{2}} \cos x\,(n-1)\sin^{n-2}x \cos x \, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,(1-\sin^2 x)\,dx$

$= (n-1)\left\{\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,dx - \int_0^{\frac{\pi}{2}} \sin^n x\,dx\right\}$

$= (n-1)\{ I_{n-2} - I_n \}$

$I_n + (n-1)I_n = (n-1)I_{n-2}$

$nI_n = (n-1)I_{n-2}$

$\therefore I_n = \frac{n-1}{n} I_{n-2}$

(ii) $I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx$

$$= \left[-\cos x \right]_0^{\frac{\pi}{2}}$$

$$= \left(-\cos \tfrac{\pi}{2}\right) - \left(-\cos 0\right)$$

$$= 1$$

$$I_3 = \frac{2}{3} \times 1 = \frac{2}{3}$$

$$I_5 = \frac{4}{5} \times I_3 = \frac{2}{3} \times \frac{4}{5}$$

$$I_7 = \frac{2 \times 4 \times 6}{3 \times 5 \quad 7}$$

$$I_9 = \frac{2 \times 4 \times 6 \times 8}{3 \times 5 \times 7 \times 9}$$

$\therefore I_n = \dfrac{2 \times 4 \times 6 \times 8 \times \ldots \times (n-1)}{1 \times 3 \times 5 \times 7 \times 9 \times \ldots \times n}$ where n is an odd positive integer

$$= \frac{2 \times 4 \times 6 \times 8 \times \ldots \times (n-1)}{1 \times 3 \times 5 \times 7 \times 9 \times \ldots \times n} \times \frac{2 \times 4 \times 6 \times 8 \times \ldots \times (n-1)}{2 \times 4 \times 6 \times 8 \times \ldots \times (n-1)}$$

$$= \frac{\left\{ 2^{(n-1)/2} \left(1 \times 2 \times 3 \times \ldots \times \frac{n-1}{2}\right) \right\}^2}{n!}$$

$$= \frac{2^{n-1} \left\{ \left(\frac{n-1}{2}\right)! \right\}^2}{n!}$$